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Modification rules for orthosymplectic superalgebras

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Abstract. Modification rules are presented for finite dimensional graded tensor representations of $\mathrm{OSp}(M/N)$. With the representations specified by Young supertableaux these rules relate non-standard supertableaux to standard supertableaux via the removal of continuous boundary hooks. All typical tensor representations are treated together with atypical representations which satisfy up to two atypicality conditions.

1. Introduction

Young tableau and Schur function techniques provide a useful and elegant description of many properties related to finite dimensional representations of semisimple Lie algebras. The extension of these techniques to Lie superalgebras (Kac 1977, 1978) was first made by Dondi and Jarvis (1980, 1981) to provide branching rules, Kronecker products and dimensions for covariant and contravariant representations of $\mathrm{U}(M/N)$ and $\mathrm{SU}(M/N)$. Since this time a number of papers have appeared (Balantekin and Bars 1981, 1982, Bars *et al* 1983, Delduc and Gourdin 1984, Morel *et al* 1984, Hurni 1984, Gourdin 1984a, b) which seek to educe properties of finite dimensional representations of Lie superalgebras in terms of supertableaux. Of particular relevance to the work presented here are the results of King (1983), which provide the branching rules

$$\mathrm{U}(M/N)\downarrow \mathrm{OSp}(M/N) \qquad \{\lambda\}\downarrow[\lambda/D] \tag{1.1}$$

$$\mathrm{OSp}(M/N)\downarrow \mathrm{O}(M)\times \mathrm{Sp}(N) \qquad [\lambda]\downarrow\sum_{\zeta}[\zeta/D]\langle\widetilde{\lambda/\zeta}\rangle \tag{1.2}$$

where ζ is any partition.

Dondi and Jarvis (1981) also pointed out that the usual Littlewood-Richardson rule for evaluating Kronecker products in $\mathrm{U}(M)$ carries over to $\mathrm{U}(M/N)$, i.e.

$$\{\lambda\}\times\{\mu\}=\{\lambda\cdot\mu\}=\sum_{\nu} m^{\nu}_{\lambda\mu}\{\nu\} \tag{1.3}$$

where $\{\lambda\}$ may be regarded as a $\mathrm{U}(M)$, $\mathrm{U}(M/N)$ or $\mathrm{SU}(M/N)$ character. King (1983) has noted that the results of Newell (1951) and Littlewood (1958) for evaluating Kronecker products of tensor representations in $\mathrm{O}(M)$ carries over to products of tensor representations in $\mathrm{OSp}(M/N)$, i.e.

$$[\lambda]\times[\mu]=\sum_{\zeta}[(\lambda/\zeta)\cdot(\mu/\zeta)] \tag{1.4}$$

where $[\lambda]$ may be regarded as an $\mathrm{O}(M)$ or an $\mathrm{OSp}(M/N)$ character.

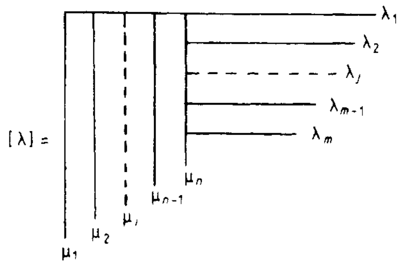
For semisimple Lie algebras a standard Young tableau possesses up to l rows, where l is the rank of the algebra being considered. The l row lengths, λ_i, $i=1,\ldots,l$, uniquely label irreducible representations of the algebra. However, when one attempts to evaluate Kronecker products of irreducible representations, e.g. as per (1.3) or (1.4), non-standard Young tableaux may arise which contain more than l rows. Non-standard tableaux also often arise in the evaluation of branching rules, as for example in (1.1), which is also the form of the branching $U(M)\downarrow O(M)$. King (1975) has given a comprehensive treatment of branching rules for classical Lie groups using Schur function techniques. For the semisimple Lie algebras the character corresponding to a non-standard tableau may either vanish or be equal, up to a sign factor, to the character corresponding to a standard tableau. The relations between irreducible representations labelled by non-standard and standard tableaux are known as *modification rules*. Early studies of these rules for the orthogonal and symplectic groups were made by Murnaghan (1938) and Newell (1951), while in more recent years King (1971) has provided a succinct procedure for modifying non-standard tableaux, which is applicable to all the classical Lie groups and is somewhat simpler to use than earlier results. King's method involves the removal of continuous boundary hooks, starting from the last box in the first column. A complete listing of the modification rules for the semisimple Lie groups and an accompanying discussion can be found in Black *et al* (1983).

For the Lie superalgebras $U(M/N)$, $SU(M/N)$ and $OSp(M/N)$, non-standard supertableaux may arise, as with Lie algebras, in the evaluation of Kronecker products (1.3), (1.4) or branching rules, e.g. (1.1). In this paper modification rules are provided for $OSp(M/N)$. These rules are in the spirit of King's method, involving the removal of continuous boundary hooks and are simple to implement. They are applicable to all typical tensor representations of $OSp(M/N)$ and atypical tensor representations which satisfy up to two atypicality conditions. This covers a large number of cases which are likely to be of interest since, as discussed in § 4, OSp(6/6) and OSp(7/6) are the lowest rank algebras for which more than two atypicality conditions can be simultaneously satisfied, with the corresponding lowest rank supertableaux being of rank 19 and 16 respectively. These rules have been deduced by decomposing partitions $[\lambda]$, of $OSp(M/N)$, according to the branching rule (1.2), where modifications can be carried out using the known results for $O(M)$ and $Sp(N)$. These decompositions have been performed with the aid of the group theory computer package SCHUR. SCHUR has enabled the examination of a large number of partitions to a maximum rank of 25, an otherwise formidable task.

The following section introduces the notation used here while §§ 3 and 4 contain the modification rules for typical and atypical supertableaux respectively.

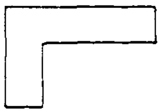
2. Notation

For $OSp(2m+1/2n)$ and $OSp(2m/2n)$ a standard Young supertableau is of the form (Farmer and Jarvis 1984)



(2.1)

where λ_j is the number of boxes beyond the nth column in the jth row, with $j\leqslant m$, and μ_i is the number of boxes in the ith column, with $i \leqslant n$. The standard $(m \times n)$ envelope of (2.1) we schematically represent by



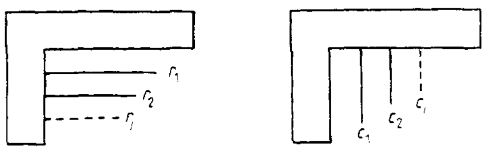
Supertableaux are always required to be regular. The row lengths λ_j, $j = 1, \ldots, m$, and column lengths μ_i, $i = 1, \ldots, n$, uniquely label tensor representations of the algebra. Under certain conditions these representations are indecomposable and are called atypical (Kac 1978). The conditions for atypicality are (Farmer and Jarvis 1984)

$$\mu_i + \lambda_j + n = i + j - 1 \tag{2.2}$$

$$\mu_i + n + j + 1 = \lambda_j + M + i \tag{2.3}$$

where $1 \leqslant i \leqslant n$; $1 \leqslant j \leqslant m$, and $M = 2m + 1$ for $\mathrm{OSp}(2m+1/2n)$ or $M = 2m$ for $\mathrm{OSp}(2m/2n)$.

Non-standard supertableaux include boxes outside the standard $(m \times n)$ envelope (2.1). These 'extra' boxes will be labelled by row lengths, r_j, or column lengths, c_i as shown below.



(2.4)

For later reference we define the partitions

$$(\hat{\lambda}) = (\lambda_1, \lambda_2, \ldots, \lambda_m) \tag{2.5}$$

$$(\hat{\mu}) = (\mu_1 - m, \mu_2 - m, \ldots, \mu_n - m) \tag{2.6}$$

where the row lengths λ_i and μ_j are taken with reference to (2.1).

A supertableau, whether it be standard or non-standard, will be called atypical if it satisfies any of the conditions (2.2) or (2.3). Otherwise it will be called typical.

In the following sections, if at any stage the modification results in an irregular supertableau then it will be discarded.

3. Typical supertableaux

Supertableaux which are typical but non-standard modify in the following way.

3.1. OSp(2m+1/2n)

(a) If $r_1 \geqslant c_1$ the modification rule is

$$[\lambda] \rightarrow [\lambda^{\mu}] = (-1)^{r-1}[\lambda - h], \qquad h = 2r_1 - 1 \tag{3.1}$$

where h is the length of the hook boundary to be removed from the partition (λ) starting from the end box in r_1 and working to the left and down, with $(r+m)$ being the row in which the removal ends.

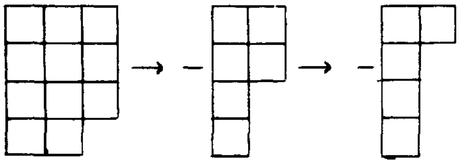
(b) If $c_1 \geqslant r_1$ the modification rule is

$$[\lambda] \rightarrow [\lambda^{\mu}] = (-1)^{c-1}[\lambda - h], \qquad h = 2c_1 - 1 \tag{3.2}$$

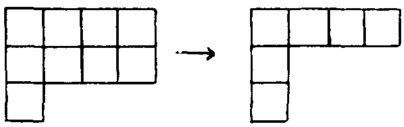
where h is the length of the hook boundary to be removed from (λ) starting from the end box in c_1 and working to the right and up, with $(c+n)$ being the column in which the removal ends.

Examples of (3.1) and (3.2) are

OSp(3/2):



OSp(3/4):



3.2. OSp(2m/2n)

(a) If $r_1 > c_1$ the modification rule is

$$[\lambda]\to[\lambda^{\mu}]=(-1)^{r}[\lambda-h], \qquad h=2r_1-2 \tag{3.3}$$

and proceed as for OSp($2m+1/2n$) case (a).

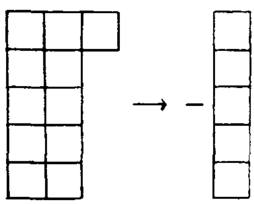
(b) If $c_1 \geqslant r_1$ the modification rule is

$$[\lambda]\to[\lambda^{\mu}]=(-1)^{c-1}[\lambda-h], \qquad h=2c_1 \tag{3.4}$$

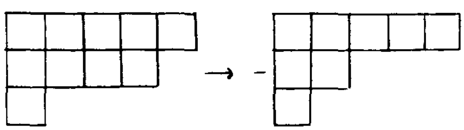
and proceed as for OSp($2m+1/2n$) case (b).

Examples of (3.3) and (3.4) are

OSp(4/2):



OSp(2/4):



These results have a natural interpretation in terms of the character formulae of King (1984). For a typical, tensor representation with corresponding standard Young supertableau $[\lambda]$, as defined by (2.1), he has noted the following:

OSp($2m+1/2n$)

$$\chi_{2m+1/2n}[\lambda]=\chi_{2m+1/2n}[n^m/E]\cdot\chi_{2m+1}[\hat{\lambda}]\cdot\chi_{2n+1}[\hat{\mu}] \tag{3.5}$$

OSp($2m/2n$)

$$\chi_{2m/2n}[\lambda]=\chi_{2m/2n}[n^m/A]\cdot\chi_{2m}[\hat{\lambda}]\cdot\chi_{2n}\langle\hat{\mu}\rangle \tag{3.6}$$

where $(\hat{\lambda})$ and $(\hat{\mu})$ are defined by (2.5) and (2.6). A definition of the infinite series of s functions A and E can be found in King (1975).

If $[\lambda']$, as defined in (2.4), is a non-standard, typical supertableau then the modification rules (3.1)-(3.4) tell us, in the light of (3.5) and (3.6), that

$$[\lambda']_{2m+1/2n}=[n^m/E]_{2m+1/2n}[\hat{\lambda}']_{2m+1}[\hat{\mu}']_{2n+1} \tag{3.7}$$

and

$$[\lambda']_{2m/2n}=[n^m/A]_{2m/2n}[\hat{\lambda}']_{2m}\langle\hat{\mu}'\rangle_{2n} \tag{3.8}$$

where if $r_1>c_1$

$$(\hat{\lambda}')=(\lambda_1, \lambda_2, \ldots, \lambda_{m-1}, \lambda_m)$$

$$(\hat{\mu}')=(\mu_1-m, \mu_2-m, \ldots, \mu_n-m, c_1, c_2, \ldots, c_i)$$

and if $c_1 \geqslant r_1$

$$(\hat{\lambda}')=(\lambda_1, \lambda_2, \ldots, \lambda_m, r_1, r_2, \ldots, r_j)$$

$$(\hat{\mu}')=(\mu_1-m, \mu_2-m, \ldots, \mu_n-m).$$

Thus the modification of a supertableau is essentially a modification of $(\hat{\lambda}')$ in $O(2m+1)$ or $O(2m)$ or of $(\hat{\mu}')$ in $O(2n+1)$ or $Sp(2n)$.

4. Atypical supertableaux

Before presenting the modification rules applicable to atypical, non-standard supertableaux, a few points need to be noted regarding which of the atypicality conditions (2.2) and (2.3) can be satisfied if the supertableaux are non-standard and, as we always require, regular.

We first note that under these conditions $\mu_i \geqslant m+1$, $1 \leqslant i \leqslant n$ and $\lambda_j \geqslant 1$, $1 \leqslant j \leqslant m$. Consequently, (2.2) can never be satisfied.

We next wish to determine which of the conditions (2.3) can be simultaneously satisfied. To do this we consider the following two expressions from (2.3)

$$\mu_a+n+b+1=\lambda_b+M+a \tag{4.1}$$

$$\mu_c+n+d+1=\lambda_d+M+c. \tag{4.2}$$

There are only two possibilities which need be examined.

(i) $a \geqslant c$ and $b<d$ or $a>c$ and $b \leqslant d$:

$$\Rightarrow \mu_c \geqslant \mu_a \tag{4.3}$$

$$\lambda_b \geqslant \lambda_d. \tag{4.4}$$

From (4.1), (4.2) and (4.3) we obtain

$$\lambda_d \geqslant \lambda_b+(a-c)+(d-b) \tag{4.5}$$

which is incompatible with (4.4).

(ii) $a \geqslant c$ and $b>d$ *or* $a>c$ and $b \geqslant d$:

$$\Rightarrow \mu_c \geqslant \mu_a \tag{4.6}$$

$$\lambda_d \geqslant \lambda_b. \tag{4.7}$$

From (4.1), (4.2) and (4.6) we obtain

$$\lambda_d \geqslant \lambda_b+(a-c)+(d-b). \tag{4.8}$$

However if $a=c$ and $b>d$ (4.8) reduces to $\lambda_d=\lambda_b+(d-b)$ which is inconsistent with (4.7); while if $b=d$ and $a>c$ (4.8) reduces to $\lambda_d=\lambda_d+(a-c)$ which is again inconsistent. Consequently (4.8) only has consistent solutions for $a>c$ and $b>d$.

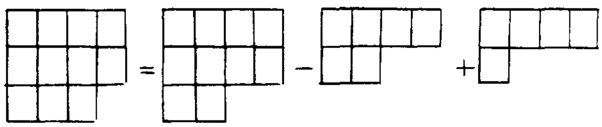
The above analysis tells us that (4.1) and (4.2) can be simultaneously satisfied iff $a > c$ and $b > d$. We will denote an atypicality condition (2.3) which relates μ_i and λ_j by (i, j). Thus if a particular (a, b) is satisfied only those conditions (i, j) for which $i > a$ and $j > b$ or $i < a$ and $j < b$ may be simultaneously satisfied. The notation $(i, j) > (k, l)$ will be used for $i > k$ and $j > l$. From this we learn that the maximum number of atypicality conditions which may be simultaneously realised is the lesser of m and n.

The modification rules for atypical supertableaux are now presented. To make their writing more succinct, h_{ij} will be used to denote a continuous boundary strip starting from the end box in column i and finishing at the end box in row j.

If the supertableau is non-standard and satisfies a *single* atypicality condition (i, j) it modifies in the following way:

$$[\lambda] = [\lambda^{\mu}] + (-1)^{\lambda_j - i - n}([\lambda - h_{ij}] - [\lambda^{\mu} - h_{ij}]) \tag{4.9}$$

where (λ^{μ}) is obtained from the modification rules of § 3 and h_{ij} is the hook boundary to be removed from both (λ) and (λ^{μ}). Any sign factors attached to $[\lambda^{\mu}]$ must be carried through for the last term in (4.9). An example of (4.9) is the following: Osp(5/4), where an atypical supertableau satisfies condition $\mu_1 = \lambda_2 + 1$.

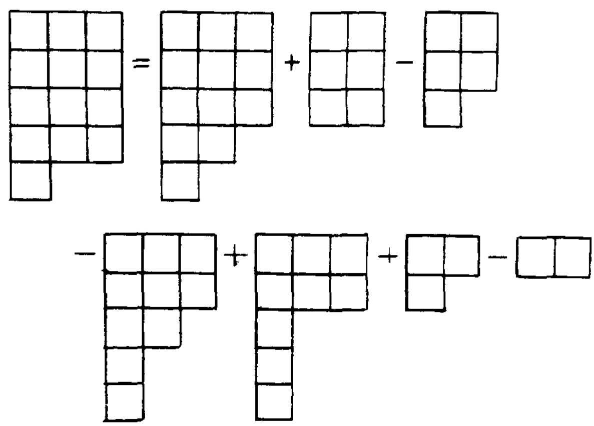


If the supertableau is non-standard and satisfies two atypicality conditions (i, j) and (k, l) with $(i, j) < (k, l)$ it modifies in the following way:

$$[\lambda] = [\lambda^{\mu}] + (-1)^{\lambda_j - i - n}([\lambda - h_{ij}] - [\lambda^{\mu} - h_{ij}]) + (-1)^{\lambda_l - k - n}([\lambda - h_{kl}] - [\lambda^{\mu} - h_{kl}])$$
$$+ (-1)^{\lambda_j + \lambda_l - i - k - 1}([\lambda - h_{kl} - h_{ij}] - [\lambda^{\mu} - h_{kl} - h_{ij}]) \tag{4.10}$$

where the hook removals in the final term must be performed in the order shown reading from left to right.

An example of (4.10) is the following: OSp(7/4), where an atypical supertableau satisfies conditions $\mu_1 = \lambda_1 + 4$ and $\mu_2 = \lambda_3 + 3$.



These rules show that for atypical supertableaux, which correspond to neither fully reducible or irreducible representations, the modification rules take on a substantially different form to those of semisimple Lie algebras. Work is currently in progress to generalise these results to include cases where an arbitrary number of atypicality conditions are simultaneously satisfied and also to derive the modification rules for spinor representations. We are also seeking to educe general proofs for the results presented here.

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